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STRESSES IN RECTANGULAR CANTILEVER CRYSTAL PLATES UNDER TRANSVERSE LOADING

P. C. Y. Lee and C. S. Lam
Department of Civil Engineering, Princeton University, Princeton, NJ 08544, U.S.A.

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Abstract—An approximate second-order theory governing the symmetric flexural vibrations (or deflection) in crystal plates is derived from the Cauchy two-dimensional flexural theory of elastic, anisotropic plates.

Close form solutions of the second-order equations of flexure are obtained for rectangular cantilever plates subject to:

(a) a uniformly distributed shear load along the edge parallel to the clamped edge, and

(b) a uniformly distributed load over the entire plate.

For load case (a), two-dimensional displacement and stresses are computed and plotted along the x_1 and x_3 axes of AT-cut plates of quartz for various values of azimuth angle ψ. As for load case (b), the solution is employed to compute the displacement and stresses in isotropic, elastic plates, so that the present results are compared with several existing results by other analytic and numerical methods.

1. INTRODUCTION

Bending of rectangular cantilever plates of isotropic elastic materials under transverse loading still remains as one of the very important and very difficult problems in the theory of elasticity. Many approximate methods have been employed to obtain two-dimensional distributions of stresses and displacement as the solutions of the Lagrange–Germain equation of flexure of elastic plates. Nash had analyzed the problem with three different approximate methods, i.e. finite-difference and two collocation methods[1]. Chang, by the concept of generalized, simply supported edge and method of superposition, solved the problem in terms of a series of infinite simultaneous equations, and had included results by Wu from finite element method for comparison[2]. Many related works to this problem can be found from the references of these two papers[1, 2].

In contrast, solutions of similar problems for anisotropic plates are scarce, and they are needed, among many important applications, for studying the effect of transverse loading and acceleration on the resonant frequencies of crystal plates[3].

In order to check the accuracy of the present solution, displacement and bending couple are computed for isotropic plates and are compared with those obtained by Nash[1], Wu and Chang[2]. Results listed in Tables 1–4 show that predictions by several different methods all agree with each other within a small range. However, we would like to note that our present approach leads to analytical solutions in closed form.

2. CAUCHY'S TWO-DIMENSIONAL FLEXURAL THEORY

Let a rectangular plate, which has a length $2a$, thickness $2b$, and width $2c$, be referred to the rectangular coordinate system (x_i), and its middle plane be coincident with the x_1x_3 plane as shown in Fig. 1. The azimuth angle ψ is the angle between the x_1 axis of the plate and the x_1' axis of the crystal. For instance, in a rotated Y-cut plate, we may choose the digonal axis of quartz as the x_1' axis. Referred to the x_i system, Cauchy's stress equation of flexural vibrations of crystal plates can be written as

$$T^{(1)}_{1,11} + 2T^{(1)}_{5,13} + T^{(1)}_{3,33} + q = 2b\rho\ddot{u}^{(0)}_2, \tag{1}$$

where $T_1^{(1)}$, $T_3^{(1)}$ are the bending couples, $T_5^{(1)}$ the twisting couple, $u_2^{(0)}$ the deflection or displacement of the plate in the x_2 direction, q the applied loading intensity (force per unit surface area) in the x_2 direction, and ρ the mass density of the crystal.

Table 1. Comparison of displacement $u_2^{(0)}$ along the free edge $x_1 = +a$ for a square plate ($r = a/c = 1$, $\nu = 0.3$)

x_3/c	0.00	0.25	0.50	0.75	1.00
Chang	$u_2^{(0)} = 0.13102 \frac{q(2a)^4}{D}$	0.13091	0.13056	0.12998	0.12933
F.E.M.	0.12905	0.12892	0.12851	0.12788	0.12708
present	0.1270	0.1268	0.1264	0.1258	0.1252

Table 2. Comparison of bending couple $T_1^{(1)}$ along the clamped edge $x_1 = -a$ for a square plate ($r = a/c = 1$, $\nu = 0.3$)

x_3/c	0.00	0.25	0.50	0.75	1.00
Chang	$T_1^{(1)} = -0.53560q(2a)^2$	-0.53550	-0.53353	-0.51270	0.0
F.E.M.	-0.53092	-0.53058	-0.52760	-0.50399	-0.34571
present	-0.5602	-0.5468	-0.5098	-0.4580	-0.4009

Table 3. Comparison of displacement $u_2^{(0)}$ along the free edge $x_1 = +a$ for a rectangular plate ($r = a/c = 0.5$, $\nu = 0.3$)

x_3/c	0.00	0.25	0.50	0.75	1.00
Chang	$u_2^{(0)} = 0.12837 \frac{q(2a)^4}{D}$	0.12825	0.12784	0.12691	0.12540
Nash	0.141		0.139		0.135
present	0.1267	0.1264	0.1256	0.1244	0.1232

Table 4. Comparison of bending couple $T_1^{(1)}$ along the clamped edge $x_1 = -a$ for a rectangular plate ($r = a/c = 0.5$, $\nu = 0.3$)

x_3/c	0.00	0.25	0.50	0.75	1.00
Chang	$T_1^{(1)} = -0.51049q(2a)^2$	-0.51451	-0.51386	-0.51074	0.0
Nash	-0.5082		-0.5047		-0.4824
present	-0.5310	-0.5241	-0.5051	-0.4784	-0.4489

The stress couples are related to the transverse shear stress resultants $T_6^{(0)}$ and $T_4^{(0)}$ by

$$T_6^{(0)} = T_{1,1}^{(1)} + T_{5,3}^{(1)}, \qquad T_4^{(0)} = T_{5,1}^{(1)} + T_{3,3}^{(1)}. \tag{2}$$

For anisotropic materials, the couple-displacement relations are

$$\begin{aligned} T_1^{(1)} &= -\frac{2b^3}{3}(\gamma_{11}u_{2,11}^{(0)} + \gamma_{13}u_{2,33}^{(0)} + 2\gamma_{15}u_{2,13}^{(0)}), \\ T_3^{(1)} &= -\frac{2b^3}{3}(\gamma_{31}u_{2,11}^{(0)} + \gamma_{33}u_{2,33}^{(0)} + 2\gamma_{35}u_{2,13}^{(0)}), \\ T_5^{(1)} &= -\frac{2b^3}{3}(\gamma_{51}u_{2,11}^{(0)} + \gamma_{53}u_{2,33}^{(0)} + 2\gamma_{55}u_{2,13}^{(0)}), \end{aligned} \tag{3}$$

where γ_{ij} are the elastic material constants. Inserting (3) into (1), we obtain

$$\gamma_{11}u_{2,1111}^{(0)} + 4\gamma_{15}u_{2,1113}^{(0)} + 2(\gamma_{13} + \gamma_{55})u_{2,1133}^{(0)} + 4\gamma_{35}u_{2,1333}^{(0)} + \gamma_{33}u_{2,3333}^{(0)} + \frac{3\rho}{b^2}\ddot{u}_2^{(0)} = \frac{3}{2b^2}q, \tag{4}$$

which is the displacement equation of flexural vibration of crystal plates.

For isotropic, elastic materials, γ_{ij} can be expressed in terms of Young's modulus E and Poisson's ratio ν by the relations

$$\gamma_{11} = \gamma_{33} = \frac{E}{1-\nu^2}, \qquad \gamma_{13} = \frac{\nu}{1-\nu^2}E, \qquad \gamma_{55} = \frac{E}{2(1+\nu)}, \qquad \gamma_{15} = \gamma_{35} = 0. \tag{5}$$

Hence, (3) are reduced to

$$\begin{aligned} T_1^{(1)} &= -D(u_{2,11}^{(0)} + \nu u_{2,33}^{(0)}) \\ T_3^{(1)} &= -D(u_{2,33}^{(0)} + \nu u_{2,11}^{(0)}) \\ T_5^{(1)} &= -D(1-\nu)u_{2,13}^{(0)} \end{aligned} \tag{3'}$$

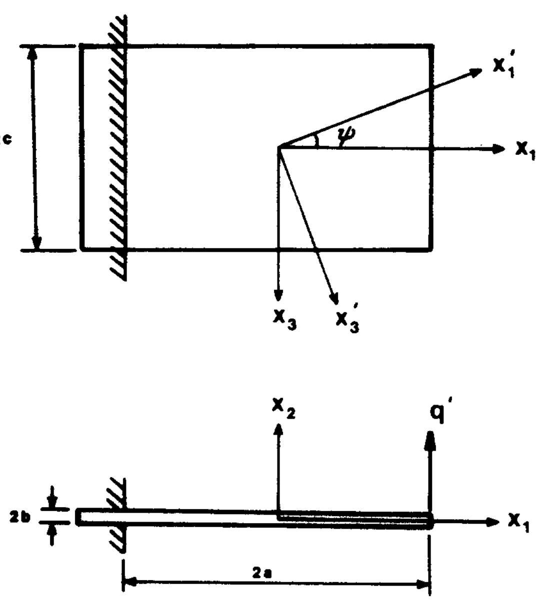

Fig. 1. A rectangular cantilever plate referred to x_i coordinate system, x_i' referred to axes of crystal symmetry, and ψ the azimuth angle.

and the displacement equation of motion (4) becomes

$$D\nabla^4 u_2^{(0)} + 2b\rho\ddot{u}_2^{(0)} = q \tag{4'}$$

for isotropic plates, where $D = (2b^3/3)(E/1-\nu^2)$ is the flexural rigidity of the plate and

$$\nabla^4 = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_3^2} + \frac{\partial^4}{\partial x_3^4}.$$

On the edges of the plate, we require that:
(1) Either $T_1^{(1)}$ or $u_{2,1}^{(0)}$ and either $T_6^{(0)} + T_{5,3}^{(1)}$ or $u_2^{(0)}$ be specified at $x_1 = \pm a$.
(2) Either $T_3^{(1)}$ or $u_{2,3}^{(0)}$ and either $T_4^{(0)} + T_{5,1}^{(1)}$ or $u_2^{(0)}$ be specified at $x_3 = \pm c$.

3. ONE-DIMENSIONAL EQUATIONS FOR FLEXURE

In order to replace approximately the two-dimensional equations (1)–(4) by a system of one-dimensional equations, we expand the displacement in a series of characteristic functions ϕ_n

$$u_2^{(0)}(x_1, x_3, t) = \sum_{n=0}^{\infty} v_2^{(n)}(x_1, t)\phi_n(x_3), \tag{6}$$

where

$$\phi_0 = 1, \qquad \phi_1 = \sqrt{3}\eta \qquad (\eta = x_3/c)$$
$$\phi_n = \frac{\cos\left(\frac{n\pi}{2} + k_n\eta\right)}{\cos\left(\frac{n\pi}{2} + k_n\right)} + \frac{\cosh\left(\frac{n\pi}{2} + ik_n\eta\right)}{\cosh\left(\frac{n\pi}{2} + ik_n\right)}, \tag{7}$$

and k_n satisfies the transcendental equations

$$\tan k_n = \pm \tanh k_n \tag{8}$$

for $n = 2, 3, 4, \ldots$. The characteristic functions $\phi_n(x_3)$ are the one-dimensional modes of (4) or (4'), when the displacement $u_2^{(0)}$ is dependent on x_3 and t only; the transcendental equation (8) derives from the traction-free conditions imposed on the edges $x_3 = \pm c$, i.e. $T_3^{(1)} = T_4^{(0)} = 0$ at $x_3 = \pm c$ (or $\eta = \pm 1$)[4]. ϕ_n and ϕ_n'' also satisfy the orthogonality conditions

$$\int_{-1}^{1} \phi_m\phi_n \, d\eta = 2\delta_{mn}, \qquad \int_{-1}^{1} \phi_m''\phi_n'' \, d\eta = 2k_n^4\delta mn. \tag{9}$$

The variational stress equation of motion corresponding to (1) is

$$\int_{t_0}^{t_1}\int_A (T_{1,11}^{(1)} + 2T_{5,13}^{(1)} + T_{3,33}^{(1)} + q - 2b\rho\ddot{u}_2^{(0)})\delta u_2^{(0)} \, dA \, dt = 0. \tag{10}$$

By substituting (6) into (10), replacing dA by $c \, dx_1 \, d\eta$, integrating by parts over $\eta = -1$ to $\eta = +1$, and requiring that the final integral be independent of any arbitrary variation $\delta v_2^{(n)}$, we obtain the nth-order stress equation of motion

$$M_{1,11}^{(n)} - \frac{2}{c}M_{5,1}^{(n)} + \frac{1}{c^2}M_3^{(n)} + q^{(n)} + V^{(n)} = 4b\rho\ddot{v}_2^{(n)}, \tag{11}$$

where

$$M_1^{(n)} = \int_{-1}^{1} T_1^{(1)}\phi_n \, d\eta,$$

$$M_5^{(n)} = \int_{-1}^{1} T_5^{(1)}\phi_n' \, d\eta,$$

$$M_3^{(n)} = \int_{-1}^{1} T_3^{(1)}\phi_n'' \, d\eta, \tag{12}$$

$$q^{(n)} = \int_{-1}^{1} q\phi_n \, d\eta,$$

$$V^{(n)} = \left[\frac{1}{c}(T_4^{(0)} + T_{5,1}^{(1)})\phi_n - \frac{1}{c^2} T_3^{(1)}\phi_n'\right]_{-1}^{1},$$

and $\phi_n' = \partial\phi/\partial\eta$, $n = 0, 1, 2, \ldots$. We note that $V^{(n)}$ is determined by the applied shear $T_4^{(0)} + T_{5,1}^{(1)}$ and the applied bending couple $T_3^{(1)}$ on the edges $x_3 = \pm c$. Therefore, for traction-free edges at $x_3 = \pm c$, $V^{(n)} = 0$.

The one-dimensional couple-displacement relations are obtained by inserting (6) into (3) and then, in turn, into (12). Thus,

$$M_1^{(m)} = -\frac{2b^3}{3}\left(2\gamma_{11}v_{2,11}^{(m)} + \frac{2}{c}\gamma_{15}\sum^{0} E_{mn'}v_{2,1}^{(n)} + \frac{1}{c^2}\gamma_{13}\sum^{e} E_{mn''}v_2^{(n)}\right),$$

$$M_5^{(m)} = -\frac{2b^3}{3}\left(\gamma_{51}\sum^{0} E_{m'n}v_{2,11}^{(n)} + \frac{2}{c}\gamma_{55}\sum^{e} E_{m'n'}v_{2,1}^{(n)} + \frac{1}{c^2}\gamma_{53}\sum^{0} E_{m'n''}v_2^{(n)}\right), \tag{13}$$

$$M_3^{(m)} = -\frac{2b^3}{3}\left(\frac{2}{c^2}\gamma_{33}k_m^4 v_2^{(m)} + \gamma_{31}\sum^{e} E_{m''n}v_{2,11}^{(n)} + \frac{2}{c}\gamma_{35}\sum^{0} E_{m''n'}v_{2,1}^{(n)}\right),$$

where

$$E_{mn'} = \int_{-1}^{1} \phi_m\phi_n' \, d\eta, \qquad E_{mn''} = \int_{-1}^{1} \phi_m\delta_n'' \, d\eta, \qquad E_{m'n'} = \int_{-1}^{1} \phi_m'\phi_n' \, d\eta, \text{ etc.}$$

$$\sum^{0} = \sum_{m+n=\text{odd}}, \qquad \sum^{e} = \sum_{m+n=\text{even}}. \tag{14}$$

We note $M_5^{(0)} = 0$, $M_3^{(0)} = M_3^{(1)} = 0$ owing to $\phi_0' = 0$ and $\phi_0'' = \phi_1'' = 0$. The values of the integration constants $E_{mn'}$, etc. in (14) and the values of k_n, roots of (8), have been studied in detail and numerical values are computed and tabulated[4]. These values are independent of material properties of the plate.

Upon substituting (13) into (1), we obtain the one-dimensional displacement equations of flexural motion

$$\gamma_{11}v_{2,1111}^{(m)} + \frac{1}{c}\gamma_{15}\sum^{0}(E_{mn'} - E_{m'n})v_{2,111}^{(n)} + \frac{1}{2c^2}\sum^{e}[\gamma_{13}(E_{mn''} + E_{m''n})$$

$$- 4\gamma_{55}E_{m'n'}]v_{2,11}^{(n)} + \frac{1}{2c^3}\gamma_{35}\sum^{0}(E_{m''n'} - E_{m'n''})v_{2,1}^{(n)} + \frac{1}{c^4}\gamma_{33}k_m^4 v_2^{(m)}$$

$$+ \frac{3}{b^2}\rho\ddot{v}_2^{(m)} = \frac{3}{4b^3}(q^{(m)} + V^{(m)}); \qquad m = 0, 1, 2, \ldots. \tag{15}$$

Equations (11), (13) and (15) form an infinite system of one-dimensional equations to formally replace the two-dimensional equations (1)–(4).

4. SECOND-ORDER EQUATIONS FOR FLEXURE

In (15), the symmetric displacement $v_2^{(n)}$, $n = 0, 2, 4, \ldots$ and the anti-symmetric displacement $v_2^{(n)}$, $n = 1, 3, 5, \ldots$ are coupled through the material constants γ_{15} and γ_{35}. In order to extract the first two equations from (15) which govern the symmetric flexural motion (or deformation) of the plate, we adopt the truncation procedure:

(1) Set $\gamma_{15} = \gamma_{35} = 0$.
(2) Retain $v_2^{(0)}$ and $v_2^{(2)}$; Let $v_2^{(n)} = 0$, $n > 2$ and $n =$ even.
(3) Set $M_1^{(n)} = M_5^{(n)} = M_3^{(n)} = 0$ for $n > 2$ and $n =$ even.

Thus we have the approximate one-dimensional second-order equations for flexural motion of crystal plates as follows.

Displacement equations of motion

$$\gamma_{11} v_{2,1111}^{(0)} + \frac{1}{2c^2}\gamma_{13}E_{02''}v_{2,11}^{(2)} + \frac{3\rho}{b^2}\ddot{v}_2^{(0)} = \frac{3}{4b^3}q^{(0)},$$

$$\gamma_{11} v_{2,1111}^{(2)} + \frac{1}{2c^2}E_{02''}\gamma_{13}v_{2,11}^{(0)} + \frac{1}{c^2}(E_{2''2}\gamma_{13} - 2E_{2'2'}\gamma_{55})v_{2,11}^{(2)} + \frac{1}{c^4}\gamma_{33}k_2^4 v_2^{(2)} + \frac{3\rho}{b^2}\ddot{v}_2^{(2)} = \frac{3}{4b^3}q^{(2)}. \tag{16}$$

Couple-displacement relations

$$M_1^{(0)} = -\frac{2b^3}{3}\left(2\gamma_{11}v_{2,11}^{(0)} + \frac{1}{c^2}\gamma_{13}E_{02''}v_2^{(2)}\right),$$

$$M_1^{(2)} = -\frac{2b^3}{3}\left(2\gamma_{11}v_{2,11}^{(2)} + \frac{1}{c^2}\gamma_{13}E_{22''}v_2^{(2)}\right),$$

$$M_5^{(2)} = -\frac{2b^3}{3}\frac{2}{c}\gamma_{55}E_{2'2'}v_{2,1}^{(2)}, \tag{17}$$

$$M_3^{(2)} = -\frac{2b^3}{3}\left(\gamma_{13}E_{2''0}v_{2,11}^{(0)} + \gamma_{13}E_{22''}v_{2,11}^{(2)} + \frac{2}{c^2}\gamma_{33}k_2^4 v_2^{(2)}\right).$$

In (16), traction-free edges at $x_3 = \pm c$ have been taken into account by setting $V^{(0)} = V^{(2)} = 0$, and in (17), we recall that $M_5^{(0)} = M_3^{(0)} = 0$. The values of k_2 and integration constants defined in (14) are [4]

$$k_2 = 2.36502, \qquad E_{02''} = E_{2''0} = 9.29455,$$
$$E_{22''} = E_{2''2} = -6.15131, \qquad E_{2'2'} = 24.7404. \tag{18}$$

At the edges $x_1 = \pm a$ of the plate, we require the specification of

(1) either $M_1^{(0)}$ or $v_{2,1}^{(0)}$,
(2) either $M_1^{(2)}$ or $v_{2,1}^{(2)}$,
(3) either $M_{1,1}^{(0)} - \frac{2}{c}M_5^{(0)}$ or $v_2^{(0)}$,
(4) either $M_{1,1}^{(2)} - \frac{2}{c}M_5^{(2)}$ or $v_2^{(2)}$. (19)

Once we have solutions $v_2^{(0)}$ and $v_2^{(2)}$ which satisfy the one-dimensional equations (16) and boundary conditions specified according to (19), we are able to obtain the two-dimensional displacement field approximately from (6)

$$u_2^{(0)}(x_1, x_3, t) = v_2^{(0)}(x_1, t) + v_2^{(2)}(x_1, t)\phi_2(x_3). \tag{20}$$

We assume that the two-dimensional bending and twisting couples are expressible by

$$T_1^{(1)}(x_1, x_3, t) = \sum_{m=1}^{\infty} H_m(x_1, t)\phi_m(x_3)$$
$$T_5^{(1)}(x_1, x_3, t) = \sum_{m=1}^{\infty} K_m(x_1, t)\phi'_m(x_3) \tag{21}$$
$$T_3^{(1)}(x_1, x_3, t) = \sum_{m=1}^{\infty} L_m(x_1, t)\phi''_m(x_3).$$

By substituting (21) into the first three equations of (12), respectively, and by employing (9) and the definition of $E_{m'n'}$ given in (14), we obtain

$$M_1^{(n)} = 2H_n, \quad M_5^{(n)} = \sum_{m=1}^{\infty} K_m E_{m'n'}, \quad M_3^{(n)} = 2k_n^4 L_n, \quad n = 1, 2, \ldots \tag{22}$$

Solve (22) for H_n, K_n and L_n and insert them into (21), then we have the two-dimensional couples for the second-order theory

$$\begin{aligned}
T_1^{(1)} &= \frac{1}{2}(M_1^{(0)} + M_1^{(2)}\phi_2)\\
&= -\frac{2b^3}{3}\left[\gamma_{11}(v_{2,11}^{(0)} + V_{2,11}^{(2)}\phi_2) + \frac{\gamma_{13}}{2c^2}(E_{02'}v_2^{(2)} + E_{22'}v_2^{(2)}\phi_2)\right],\\
T_5^{(1)} &= \frac{1}{E_{2'2'}} M_5^{(2)}\phi_2'\\
&= -\frac{2b^3}{3}\frac{2\gamma_{55}}{c} v_{2,1}^{(2)}\phi_2,\\
T_3^{(1)} &= \frac{1}{2k_2^4} M_3^{(2)}\phi_2''\\
&= -\frac{2b^3}{3}\left[\frac{\gamma_{13}}{2k_2^4}(E_{02'}v_{2,11}^{(0)} + E_{22'}v_{2,11}^{(2)}) + \frac{\gamma_{33}}{c^2} v_2^{(2)}\right]\phi_2''.
\end{aligned} \tag{23}$$

Then, the two-dimensional transverse shear resultants $T_6^{(0)}$ and $T_4^{(0)}$ can be obtained by substitution of (23) into (2).

5. CANTILEVER RECTANGULAR PLATES SUBJECT TO TRANSVERSE LOADINGS

Closed form solutions of (16) are obtained for cantilever crystal plates subject to two kinds of transverse loading:

(a) Transverse shear loading distributed uniformly over the plate edge parallel to the support.

(b) Uniformly distributed transverse loading over the entire plate.

Since both problems (a) and (b) are static, $\ddot{v}_2^{(0)} = \ddot{v}_2^{(2)} = 0$, and (16) may be written

$$\begin{aligned}
c^4 v_{2,1111}^{(0)} + c^2 Q_2 v_{2,11}^{(2)} &= F_0,\\
c^4 v_{2,1111}^{(2)} + c^2 Q_1 v_{2,11}^{(2)} + c^2 Q_2 v_{2,11}^{(0)} + Q_3 v_2^{(2)} &= F_2,
\end{aligned} \tag{24}$$

where

$$Q_1 = \bar{\gamma}_{13}E_{22'} - 2\bar{\gamma}_{55}E_{2'2'}, \quad Q_2 = \frac{1}{2}\bar{\gamma}_{13}E_{02'}, \quad Q_3 = \bar{\gamma}_{33}k_2^4,$$
$$\bar{\gamma}_{ij} = \gamma_{ij}/\gamma_{11}, \quad F_0 = \frac{3c^4 q^{(0)}}{4b^3\gamma_{11}}, \quad F_2 = \frac{3c^4 q^{(2)}}{4b^3\gamma_{11}}. \tag{25}$$

Problem (a)

Since there is no applied load over the plate, q must be zero. Then from the fourth equation of (12), we have $q^{(0)} = q^{(2)} = 0$ and from (25), $F_0 = F_2 = 0$ in (24).

Let the uniform shear loading intensity at edge $x_1 = +a$ be q', which has the dimension of force per unit length. Then for the cantilever plate with edge shear, we require, according to (19),

$$v_2^{(0)} = v_2^{(2)} = v_{2,1}^{(0)} = v_{2,1}^{(2)} = 0 \qquad \text{at } x_1 = -a$$

$$M_1^{(0)} = M_1^{(2)} = 0, \qquad M_{1,1}^{(0)} = 2q', \qquad M_{1,1}^{(2)} - \frac{2}{c} M_5^{(2)} = 0, \qquad \text{at } x_1 = +a. \tag{26}$$

We assume the solution form

$$v_2^{(0)} = A\, e^{\lambda x_1/c}, \qquad v_2^{(2)} = G\, e^{\lambda x_1/c}. \tag{27}$$

Equations (16) are satisfied by (27), provided

$$\begin{bmatrix} \lambda^4 & Q_2\lambda^2 \\ Q_2\lambda^2 & \lambda^4 + Q_1\lambda^2 + Q_3 \end{bmatrix} \begin{bmatrix} A \\ G \end{bmatrix} = 0. \tag{28}$$

The vanishing of the determinant of the coefficient matrix of (28) leads to four repeated zero roots of λ and four non-zero roots λ_p, $p = 1, 2, 3, 4$ which satisfy

$$\lambda_p^4 + Q_1\lambda_p^2 + (Q_3 - Q_2^2) = 0; \qquad p = 1, 2, 3, 4. \tag{29}$$

Therefore, we may express the general solution by

$$v_2^{(0)} = \sum_{p=1}^{4} A_p\, e^{\lambda_p r \bar{x}_1} + A_5\bar{x}_1^3 + A_6\bar{x}_1^2 + A_7\bar{x}_1 + A_8,$$

$$v_2^{(2)} = \sum_{p=1}^{4} \alpha_p A_p\, e^{\lambda_p r \bar{x}_1} + \alpha_5 A_5 \bar{x}_1 + \alpha_5 A_6, \tag{30}$$

where

$$\alpha_p = \frac{G_p}{A_p} = -\frac{\lambda_p^2}{Q_2}; \qquad p = 1, 2, 3, 4$$

$$\alpha_5 = -\frac{6Q_2}{Q_3} r^{-2}, \qquad \alpha_6 = -\frac{2Q_2}{Q_3} r^{-2},$$

$$\bar{x}_1 = x_1/a, \qquad r = a/c. \tag{31}$$

Substitution of (30) into (26), with the aid of (17), leads to a system of eight non-homogeneous equations for the eight amplitudes A_j, $j = 1, 2, \ldots, 8$, which may be written in the matrix notation

$$[M_{ij}][A_j] = [B_i]; \qquad i, j = 1, 2, 3, \ldots, 8, \tag{32}$$

in which the non-zero elements are given below.

$$M_{1q} = e^{-\lambda_q r}, \qquad M_{2q} = \alpha_q\, e^{-\lambda_q r}, \qquad M_{3q} = \lambda_q\, e^{-\lambda_q r}, \qquad M_{4q} = \alpha_q\lambda_q\, e^{-\lambda_q r},$$

$$M_{5q} = (2\lambda_q^2 + \bar{\gamma}_{13}E_{02''}\alpha_q)\, e^{\lambda_q r}, \qquad M_{6q} = (2\alpha_q\lambda_q^2 + \bar{\gamma}_{13}E_{22''}\alpha_q)\, e^{\lambda_q r}, \qquad M_{7q} = (2\lambda_q^3 + \bar{\gamma}_{13}E_{02''}\alpha_q\lambda_q)\, e^{\lambda_q r},$$

$$M_{8q} = [2\alpha_q\lambda_q^3 + (\bar{\gamma}_{13}E_{22''} - 4\bar{\gamma}_{55}E_{2'2'})\alpha_q\lambda_q]\, e^{\lambda_q r}, \qquad M_{15} = -1, \qquad M_{16} = 1, \qquad M_{17} = -1, \qquad M_{18} = 1,$$

$$M_{25} = -\alpha_5, \qquad M_{26} = \alpha_6, \qquad M_{35} = 3r^{-1}, \qquad M_{36} = -2r^{-1}, \qquad M_{37} = r^{-1}, \qquad M_{45} = \alpha_5 r^{-1},$$

$$M_{55} = (12r^{-2} + \bar{\gamma}_{13}E_{02''}\alpha_5), \qquad M_{56} = (4r^{-1} + \bar{\gamma}_{13}E_{02''}\alpha_6), \qquad M_{65} = \bar{\gamma}_{13}E_{22''}\alpha_5.$$

$$M_{66} = \bar{\gamma}_{13}E_{22''}\alpha_6, \qquad M_{75} = (12r^{-3} + \bar{\gamma}_{13}E_{02''}\alpha_5 r^{-1}), \qquad M_{85} = (\bar{\gamma}_{13}E_{22''} - 4\bar{\gamma}_{55}E_{2'2'})\alpha_5 r^{-1},$$

$$B_7 = -3q'c^3/\gamma_{11}b^3; \qquad q = 1, 2, 3, 4. \tag{33}$$

A rectangular AT-cut quartz plate with $r = a/c = 1.50$, shown in Fig. 1, is chosen for the study. Let the x_1' axis be the diagonal axis of quartz. The material constants γ_{ij} referred to the plate axes x_i must be functions of the azimuth angle ψ according to the rule of tensor transformation between c_{ij} and c'_{ij} of the elastic stiffness coefficients of quartz.

For a given edge shear intensity q', amplitudes A_j can be calculated from (32), and $v_2^{(0)}$ and $v_2^{(2)}$ from (30). Then the two-dimensional displacement $u_2^{(0)}$ is obtained from (20), and stress couples and stress resultants from (23) and (2), respectively, with ψ regarded as a parameter.

For the visualization of their two dimensional variations in the plate, the displacement and stresses are computed and plotted along the x_1 and x_3 axes in dimensionless quantities and for various values of ψ.

The variation of displacement $u_2^{(0)}$ along the x_1 and x_3 axes is shown in Figs. 2 and 3, respectively. In a similar manner, the stress couple $T_1^{(1)}$ is shown in Figs. 4 and 5, $T_3^{(1)}$ in Figs. 6 and 7, $T_5^{(1)}$ in Fig. 8, shear stress resultant $T_6^{(0)}$ in Figs. 9 and 10, and $T_4^{(0)}$ in Fig. 11. We note that both $T_5^{(1)}$ and $T_4^{(0)}$ vanish along the x_1 axis.

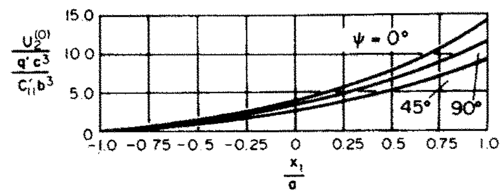

Fig. 2. Displacement $u_2^{(0)}$ along the x_1 axis of a rectangular AT-cur of quartz plate ($r = a/c = 1.50$).

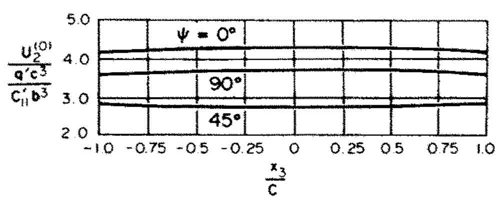

Fig. 3. Displacement $u_2^{(0)}$ along the x_3 axis of the plate.

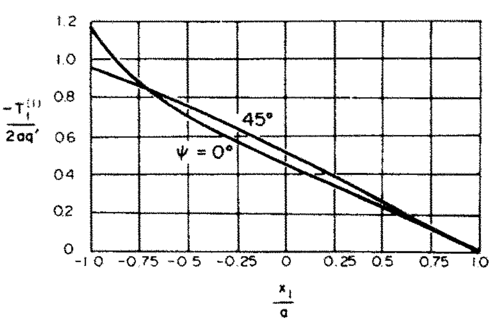

Fig. 4. Bending couple $T_1^{(1)}$ along the x_1 axis of the plate.

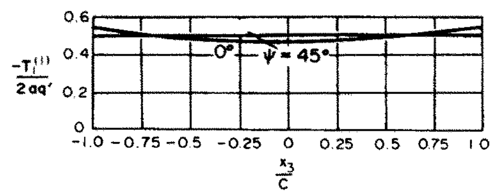

Fig. 5. Bending couple $T_1^{(1)}$ along the x_3 axis of the plate.

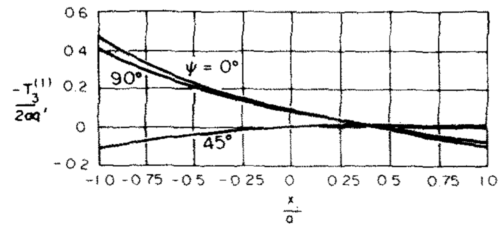

Fig. 6. Bending couple $T_3^{(1)}$ along the x_1 axis of the plate.

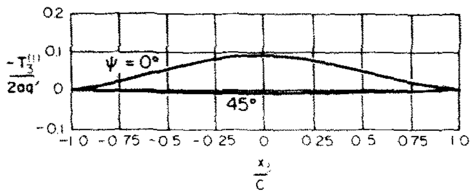

Fig. 7. Bending couple $T_3^{(1)}$ along the x_3 axis of the plate.

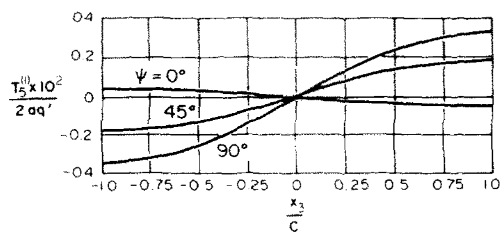

Fig. 8. Twisting couple $T_5^{(1)}$ along the x_3 axis of the plate.

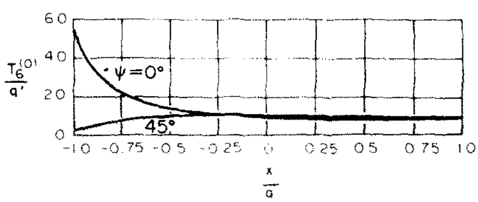

Fig. 9. Transverse shear $T_6^{(0)}$ along the x_1 axis of the plate.

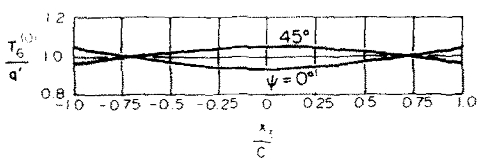

Fig. 10. Transverse shear $T_6^{(0)}$ along the x_3 axis of the plate.

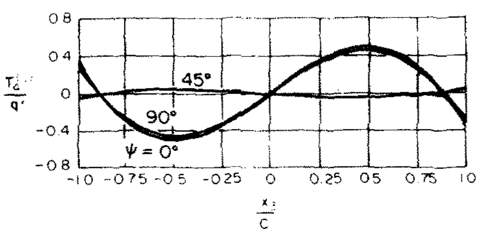

Fig. 11. Transverse shear $T_4^{(0)}$ along the x_3 axis of the plate.

It may be seen in Figs. 2 and 3 that the middle plane of the deformed plate is convex upward in the lengthwise direction, while the curvature in the crosswise direction is convex downward for $\psi = 0$. This effect is called *anticlastic* and is very similar to that for isotropic plates. However, the curvature in the crosswise direction for $\psi = 45°$, in Fig. 3, is convex upward. This results from the fact that, owing to the anisotropy of the plate, the γ_{ij} may change both their magnitude and sign for different angle ψ for different plate orientations. In this case of the *AT*-cut of quartz plate, γ_{13}, indeed, changes its value from $27.56 \times 10^9\,\mathrm{N/cm^2}$ for $\psi = 0°$ to $-8.41 \times 10^9\,\mathrm{N/cm^2}$ for $\psi = 45°$ which accounts for the corresponding changes in the crosswise curvature.

It is of interest to compare the distributions of couples of shears in Figs. 4–11 with those from the one-dimensional solution of a cantilever beam. Corresponding to beam solutions, $T_1^{(1)}$ would be linearly proportional to x_1/a in Fig. 4 and be constant crosswise in Fig. 5; $T_6^{(0)}$ in Figs. 8 and 9 would be constant both lengthwise and crosswise, and $T_3^{(1)}$, $T_5^{(1)}$ and $T_4^{(0)}$ would be zero throughout the plate.

Problem (*b*)

When the plate is subject to a uniform load over the entire surface, the loading intensity q becomes a constant. Thus, from (12) we get

$$q^{(0)} = 2q, \qquad q^{(2)} = 0, \tag{34}$$

for $\int_{-1}^{1} \phi_2 \,\mathrm{d}\eta = 0$. Therefore in (24) and (25), we have

$$F_0 = \frac{3c^4 q}{2b^3 \gamma_{11}}, \qquad F_2 = 0. \tag{35}$$

Since there is no applied edge force, the boundary conditions (26) reduce to

$$\begin{aligned} &v_2^{(0)} = v_2^{(2)} = v_{2,1}^{(0)} = v_{2,1}^{(2)} = 0 \qquad \text{at } x_1 = -a \\ &M_1^{(0)} = M_1^{(2)} = M_{1,1}^{(0)} = M_{1,1}^{(2)} - \frac{2}{c} M_5^{(2)} = 0 \qquad \text{at } x_1 = +a. \end{aligned} \tag{36}$$

We may specify that the solution of (24) in this case is the sum of the homogeneous solution (when $F_0 = F_2 = 0$) and a particular solution for F_0, F_2 given in (35). The homogeneous solution is identical to (31). The particular solution is obtained in a polynomial form

$$\bar{v}_2^{(0)} = F_0 P_1 r^4 \bar{x}_1^{\,4}, \qquad \bar{v}_2^{(2)} = F_0 (P_2 r^2 \bar{x}_1^{\,2} + P_3), \tag{37}$$

where

$$P_1 = -\frac{Q_3}{24(Q_2^2 - Q_3)}, \qquad P_2 = \frac{Q_2}{2(Q_2^2 - Q_3)}, \qquad P_3 = \frac{-Q_1 Q_2}{Q_3(Q_2^2 - Q_3)}. \tag{38}$$

We note that the particular solution is completely determined by the applied load F_0 and the coefficients of the governing equation Q_1, Q_2 and Q_3 defined in (25).

By adding (37) to (31), we have the complete solution. Thus,

$$\begin{aligned} v_2^{(0)} &= F_0 \left(\sum_{p=1}^{4} A_p \,\mathrm{e}^{\lambda_p r \bar{x}_1} + A_5 \bar{x}_1^{\,3} + A_6 \bar{x}_1^{\,2} + A_7 \bar{x}_1 + A_8 + P_1 r^4 \bar{x}_1^{\,4} \right), \\ v_2^{(2)} &= F_0 \left(\sum_{p=1}^{4} \alpha_p A_p \,\mathrm{e}^{\lambda_p r \bar{x}_1} + \alpha_5 A_5 \bar{x}_1 + \alpha_6 A_6 + P_2 r^2 \bar{x}_1^{\,2} + P_3 \right). \end{aligned} \tag{39}$$

Again, the requirement of boundary conditions (36) results in a system of eight equations on eight unknowns A_j, $j = 1, 2, 3, \ldots, 8$

$$[M_{ij}][A_j] = [B_i'] \tag{40}$$

where the elements M_{ij} are identical to those defined in (33), while the elements B'_i are given as follows.

$$B'_1 = -P_1 r^4, \qquad B'_2 = -P_2 r^2 - P_3, \qquad B'_3 = 4P_1 r^3, \qquad B'_4 = 2P_2 r,$$
$$B'_5 = -24P_1 r^2 - \bar{\gamma}_{13} E_{02''}(P_2 r^2 + P_3), \qquad B'_6 = -4P_2 - \bar{\gamma}_{13} E_{22''}(P_2 r^2 + P_3), \tag{41}$$
$$B'_7 = -48P_1 r - 2\bar{\gamma}_{13} E_{02''} P_2 r, \qquad B'_8 = (4\bar{\gamma}_{55} E_{2'2'} - \bar{\gamma}_{13} E_{22''}) P_2 r.$$

We see that (32) and (41) have identical form; they differ only in B_j and B'_j.

Once the A_j are calculated from (40), $v_2^{(0)}$ and $v_2^{(2)}$ can be obtained from (39). Then the two-dimensional displacement $u_2^{(0)}$ can be calculated from (20), and stresses from (23) and (2).

To check the accuracy of our solution, we have computed displacements and stresses for isotropic elastic plates by converting the material constants through (5), since many existing solutions by various different methods of approximation are available only for isotropic plates.

First, a square plate ($r = a/c = 1$) with Poisson's ratio $\nu = 0.3$ is chosen. The displacement $u_2^{(0)}$ along the free edge $x_1 = a$ and the bending stress $T_1^{(1)}$ along the clamped edge $x_1 = -a$ are computed as functions of x_3/c, and they are listed in Tables 1 and 2, respectively, together with the results by Chang[2] and by Wu (F.E.M.)[2].

A second comparison is made for a rectangular plate ($r = a/c = 0.5$) and $\nu = 0.3$. In a similar manner, the present results are compared with those by Nash[1] and Chang[2] as shown in Tables 3 and 4.

It can be seen that the agreement among the results by various methods is close. The predicted displacement by the present method is about 1.5% lower than those by the finite element method for the square plate (Table 1), and 1.7% lower than the values by Chang for the rectangular plate (Table 3). The value of the bending couple at the middle point in the clamped edge ($x_1/a = -1$, $x_3/c = 0$) is about $3 \sim 5\%$ higher than those predicted by Chang, Nash and Wu. At the outer points of the clamped edge ($x_1/a = -1$, $x_3/c = \pm 1$), only Chang's predicted bending couple $T_1^{(1)}$ has the exact value, i.e. $T_1^{(1)} = 0$.

In summary, the present method of approach offers closed form solutions which give excellent agreement in predicted displacement, and the method can be easily applied to both the isotropic and anisotropic plates.

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